Mode-Matching Analysis of the Step Discontinuity in Elliptical Waveguides

Pawel Matras, Rainer Bunger, and Fritz Arndt, *Fellow, IEEE*

Abstract— **The modal scattering matrix of the step discontinuity of two elliptical waveguides of different cross sections is calculated rigorously by the direct mode-matching method using the Mathieu equation. For the convenient treatment of the Mathieu functions, an efficient trigonometric series expansion technique is used. As examples, the input reflection coefficients are calculated of two step discontinuities, a nearly circular-to-circular waveguide transition and a transition from larger to smaller confocal elliptical waveguide. Excellent agreement with reference results verifies the accuracy of the presented method.**

I. Introduction

Elliptical waveguides [1]–[7] have found increasing application in the design of many microwave structures with specific characteristics, such as radiators [5], corrugated horns [4], and resonators [7]. Although the cross-section eigenvalue problem has already been solved by many authors for a rather long time [1]–[3] and has attracted recent interest [6]–[8], the three-dimensional (3-D) scattering problem of related structures has not yet been treated by the mode-matching method so far. The availability of fast and efficient methods for deriving the modal scattering parameters, however, is important for the reliable design and optimization of elliptical waveguide structures with improved performance.

This letter describes the rigorous direct mode-matching analysis of the scattering problem at the discontinuity of two elliptical waveguides of different cross section (Fig. 1). The solution of the related eigenvalue (Mathieu) equation is advantageously based on an efficient trigonometric expansion technique.

II. Theory

For a waveguide of elliptical cross section (see Fig. 1) with the focal distance $2h$, the wave equation for the corresponding transversal eigenfunctions $T(\xi, \eta) = U(\xi)V(\eta)$ is given in elliptic coordinates ξ, η, z by [6]

$$\frac{\partial^2 V}{\partial \eta^2} + (a - 2q\cos 2\eta)V = 0$$
$$\frac{\partial^2 U}{\partial \xi^2} - (a - 2q\cosh 2\xi)U = 0 \quad (1)$$

with the abbreviation

$$q = \frac{k_c^2}{4} h^2 \quad (2)$$

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The authors are with the Microwave Department, University of Bremen, D-28359, Bremen, Germany.
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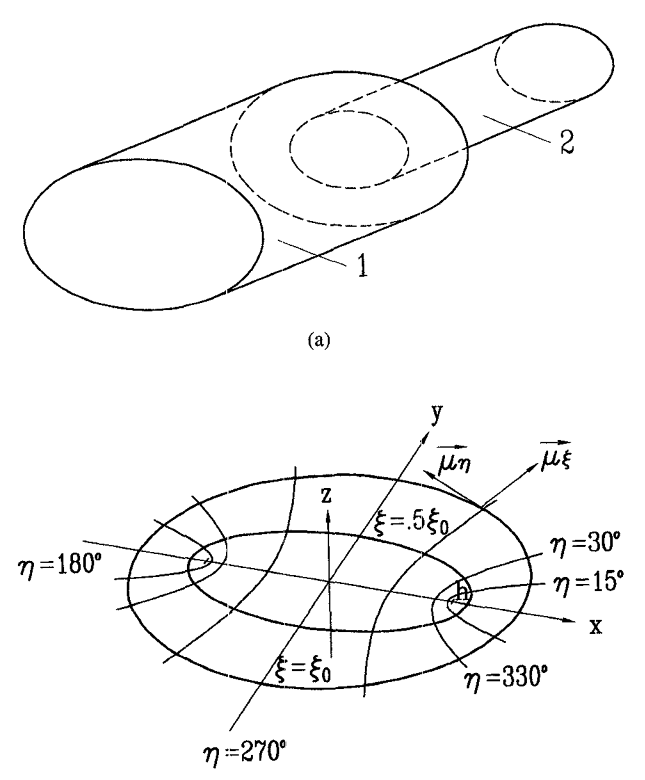

Fig. 1. Transition larger to smaller elliptical waveguide: (a) step discontinuity and (b) elliptical coordinate system.

where k_c is the cut-off wavenumber and a is a constant, denoted as the "separation constant" [6]. The second equation of (1), the modified Mathieu equation, is related to the first equation—the Mathieu equation—via the transformation $\xi = j\eta$, where $j = \sqrt{-1}$.

The formal solution of the eigenvalue problem, (1), leads to [6]

$$T(\xi, \eta) = \begin{Bmatrix} \mathrm{Ce}(\xi)\,\mathrm{ce}(\eta) \\ \mathrm{Se}(\xi)\,\mathrm{se}(\eta) \end{Bmatrix} \quad (3)$$

where ce, se, and Ce, Se denote the even and odd Mathieu and modified even and odd Mathieu functions, respectively.

Equation (1) is solved numerically very efficiently by the trigonometric series expansion of [9] and [10][1]

$$\mathrm{ce}_{2n}(\eta) = \sum_{r=0}^{\infty} A_{2r}\cos(2r\eta)$$

[1]Here, the standard notation of [10] is used. The indexes n represent the dependency of the coefficients A and B on n.

$$\mathrm{ce}_{2n+1}(\eta) = \sum_{r=0}^{\infty} A_{2r+1}\cos[(2r+1)\eta] \tag{4}$$

$$\mathrm{se}_{2n+1}(\eta) = \sum_{r=0}^{\infty} B_{2r+1}\sin[(2r+1)\eta]$$

$$\mathrm{se}_{2n+2}(\eta) = \sum_{r=0}^{\infty} B_{2r+2}\sin[(2r+2)\eta]. \tag{5}$$

With (1)–(5), first the separation constant a in (1), and then the coefficients in (4) and (5) are determined numerically. For a, the following equation is obtained

$$a - m^2 - u_m - v_m = 0 \quad m \geq 0 \tag{6}$$

with the abbreviations[2]

$$m = \begin{cases} 2n, & \text{for } \mathrm{ce}_{2n} \\ 2n+1, & \text{for } \mathrm{ce}_{2n+1} \text{ and } \mathrm{se}_{2n+1} \\ 2n+2, & \text{for } \mathrm{se}_{2n+2} \end{cases}$$

$$v_m = \frac{t_v q^2}{a-(m+2)^2 - v_{m+2}}, \qquad n \geq 0$$

$$u_m = \begin{cases} u_0 & n = 0 \\ \dfrac{t_u q^2}{a-(m-2)^2 - u_{m-2}}, & n \geq 1 \end{cases} \tag{7}$$

where

$$t_v = \begin{cases} 2, & \text{for ce and } v_0 \\ 1, & \text{else} \end{cases}$$

$$t_u = \begin{cases} 2, & \text{for ce and } u_2 \\ 1, & \text{else} \end{cases} \tag{8}$$

and

$$u_0 = \begin{cases} 0, & \text{for even } m \\ q, & \text{for odd } m \text{ and ce} \\ -q, & \text{for odd } m \text{ and se.} \end{cases}$$

v and u are supplementary variables [10] formulated by the ratio of the corresponding coefficients in (4) and (5), e.g., for the first equation in (4)

$$v_i = \frac{A_{i+2}}{A_i}$$

$$u_i = \frac{A_{i-2}}{A_i}. \tag{9}$$

The consideration of the normalization conditions [10], e.g., for $\mathrm{ce}_{2n}(\eta)$

$$\int_0^{2\pi} \left[\sum_{r=0}^{\infty} A_{2r}\cos(2rz)\right]^2 dz \overset{!}{=} \pi \tag{10}$$

leads to expressions of the form

$$2 + v_0^2 + (v_0 v_2)^2 + (v_0 v_2 v_4)^2 + \cdots = \frac{1}{A_0^2}. \tag{11}$$

[2]This notation, different from that in [10], is common for all modes. This leads merely to one equation (instead of four); the parameters need to be initialized only when the mode changes (t_u, t_v) or at the beginning of the calculation of the value of the functions (v_m, u_m).

From (11) A_0 can be determined. Generally, the first coefficient can be calculated by

$$C = \left| \sqrt{\frac{1}{t_C + \sum_{r=1}^{\infty}\left(\prod_{n=0}^{r-1} v_m\right)^2}} \right| \tag{12}$$

where

$$t_C = \begin{cases} 2, & \text{for } \mathrm{ce}_{2n} \\ 1, & \text{else} \end{cases}$$

$$C = \begin{cases} A_0, & \text{for } \mathrm{ce}_{2n} \\ A_1, & \text{for } \mathrm{ce}_{2n+1} \\ B_2, & \text{for } \mathrm{se}_{2n+2} \\ B_1, & \text{for } \mathrm{se}_{2n+1}. \end{cases}$$

The other coefficients A or B are then calculated by (7) and (9). (For the calculation of B, in (9) A should be replaced by B.)

The cut-off frequencies of typical elliptical waveguide cross sections have been calculated up to 300 higher-order modes and have been compared with own FEM calculations and values reported in [8]. Excellent accuracy (up to eight digits) has been obtained.

The modal scattering matrix of the discontinuity (Fig. 1) is obtained in the usual form [11] by matching of the tangential field components. Application of the orthogonality of the eigenfunctions and rearranging the equations yields the modal scattering matrix of the discontinuity directly

$$\mathbf{a}_1 + \mathbf{b}_1 = [\mathbf{M}](\mathbf{a}_2 + \mathbf{b}_2), \quad \mathbf{a}_2 - \mathbf{b}_2 = [\mathbf{M}]^T(\mathbf{a}_1 - \mathbf{b}_1) \tag{13}$$

where[3]

$$\mathbf{M} = \boldsymbol{\delta}_1 \mathbf{K} \boldsymbol{\delta}_2 \tag{14}$$

with the diagonal matrices $\boldsymbol{\delta}$ containing the normalization expressions N and the frequency-dependent wave impedances and admittances of the adjacent waveguides, respectively. These results are valid for the confocal and nonconfocal case. Since the explicit presentation of the coupling integrals for the general case would be beyond the scope of this letter, only the frequency-independent coupling integrals of a discontinuity of confocal elliptic waveguides are given.

1) For the TE-TE coupling of the waves $\mathrm{TE}_{Cm,i}$ and $\mathrm{TE}_{Cn,j}$, or $\mathrm{TE}_{Sm,i}$ and $\mathrm{TE}_{Sn,j}$

$$K_{k,l} = \frac{\pi q_{c2}^2}{q_{c2}^2 - q_{c1}^2} \begin{Bmatrix} \mathrm{Ce}_2(\xi_0)\,\mathrm{Ce}_2'(\xi_0) \\ \mathrm{Se}_2(\xi_0)\,\mathrm{Se}_1'(\xi_0) \end{Bmatrix} \sum_{r=0}^{\infty} \begin{Bmatrix} A_{pc2}A_{pc1} \\ A_{ps2}A_{ps1} \end{Bmatrix}. \tag{15}$$

2) For the TM-TM coupling of the waves $\mathrm{TM}_{Cm,i}$ and $\mathrm{TM}_{Cn,j}$, or $\mathrm{TM}_{Sm,i}$ and $\mathrm{TM}_{Sn,j}$

$$K_{k,l} = \frac{\pi q_{c1}^2}{q_{c1}^2 - q_{c2}^2} \begin{Bmatrix} \mathrm{Ce}_1(\xi_0)\,\mathrm{Ce}_2'(\xi_0) \\ \mathrm{Se}_1(\xi_0)\mathrm{Se}_2'(\xi_0) \end{Bmatrix} \sum_{r=0}^{\infty} \begin{Bmatrix} A_{pc2}A_{pc1} \\ A_{ps2}A_{ps1} \end{Bmatrix}. \tag{16}$$

[3]The index "1" denotes the larger waveguide, and the index "2" denotes the smaller one.

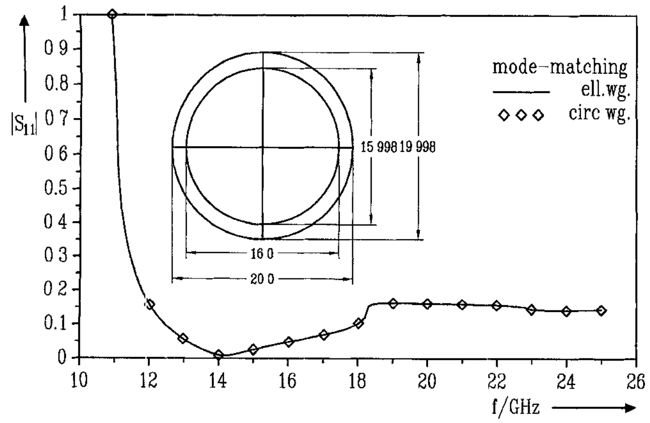

Fig. 2. Input reflection coefficient as a function of frequency for a discontinuity of elliptical waveguides with nearly circular geometry. (Reference results are for exactly circular waveguides.)

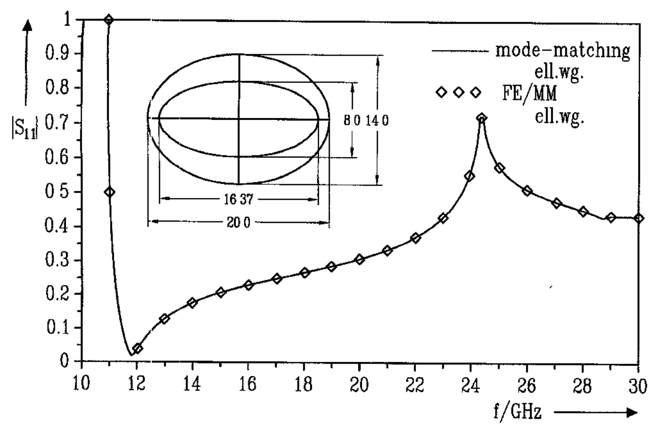

Fig. 3. Input reflection coefficient as a function of frequency for the discontinuity of two confocal elliptic waveguides.

3) For the TM-TE coupling of the waves $TM_{Cm,i}$ and $TE_{Sn,j}$

$$K_{k,l} = \pi \operatorname{Ce}_1(\xi_0) \operatorname{Se}_2(\xi_0) \sum_{r=0}^{\infty} pA_{pc1}A_{ps2}. \qquad (17)$$

4) For the TM-TE coupling of the waves $TM_{Sm,i}$ and $TE_{Cn,j}$

$$K_{k,l} = -\pi \operatorname{Se}_1(\xi_0) \operatorname{Ce}_2(\xi_0) \sum_{r=0}^{\infty} pA_{pc2}A_{ps1}. \qquad (18)$$

q is related to the cut-off wavenumber by (2), the dash denotes the corresponding derivative, and p is the abbreviation

$$p = \begin{cases} 2r & r = 1, 2, 3, \cdots \text{ for both even modes} \\ 2r+1 & r = 0, 1, 2, \cdots \text{ for both odd modes.} \end{cases}$$

For all other couplings, the coupling integrals are zero, particularly those between odd and even wave modes.

III. Results

Fig. 2 shows the input reflection coefficient as a function of frequency for a discontinuity of elliptical waveguides with nearly circular geometry. Excellent agreement with reference values obtained by the mode-matching method for circular waveguides is obtained. In Fig. 3, the input reflection coefficient for the discontinuity of two confocal elliptic waveguides is calculated and verified with values obtained by the FE/MM method [11]. For the calculations, all higher-order modes in the order of increasing cut-off are considered up to the cut-off frequency of 100 GHz. The efficiency of the presented method is demonstrated by the fact that the above direct mode-matching results are calculated by using a simple 486-level PC, where the cpu time for a typical frequency response was less than 20 min.

IV. Conclusion

A direct mode-matching technique is proposed for the calculation of the modal scattering matrix of the confocal step discontinuity of elliptical waveguides. Because of the high numerical efficiency of the method, only a standard PC is required for the rigorous analysis of the investigated discontinuities.

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